Math 564: Adv. Analysis 1 HOMEWORK 5 **Due: Nov 21 (Tue), 11:59pm**

1. Consider $\mathbb{R}^d$ with Lebesgue measure λ and let $L^1 := L^1(\mathbb{R}^d, \lambda)$.

(a) Prove that for every $f \in L^1$ and $\varepsilon > 0$, there is a simple function s that is a linear combination of indicator functions of bounded boxes such that $\|f - s\|_1 < \varepsilon$.

HINT: Firstly, make things bounded by noting that $\|f - f\mathbb{1}_{B_N}\|_1 < \varepsilon/2$ for all large enough $N \in \mathbb{N}$, where B_N is the cube of side-length N centered at the origin.

(b) Prove that for every bounded box $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, there is a continuous function $g_B : \mathbb{R}^d \to \mathbb{R}$ with support $\subseteq B$ such that $\|\mathbb{1}_B - g_B\|_1 < \varepsilon$.

HINT: Do this for $d = 1$ first.

(c) Deduce that for every $f \in L^1$ and $\varepsilon > 0$, there is a continuous function $g : \mathbb{R}^d \to \mathbb{R}$ of bounded support such that $\|f - g\|_1 < \varepsilon$. In other words, continuous functions (of bounded support) are dense in L^1.

2. Let $f : (0, \infty) \to \mathbb{R}$ be a Lebesgue integrable function. Prove:

(a) $g(x) := \int_x^\infty t^{-1} f(t) d\lambda(t)$ is well-defined for each $x > 0$, i.e. $t \mapsto \mathbb{1}_{(x,\infty)}(t) t^{-1} f(t)$ is a Lebesgue integrable function.

(b) The function $g : (0, \infty) \to \mathbb{R}$ is Lebesgue integrable and

$$\int_0^\infty g \, d\lambda = \int_0^\infty f \, d\lambda.$$

3. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{B})$.

(a) Prove the **Lebesgue decomposition theorem** directly, without using signed measures: there is a partition $X = X_0 \sqcup X_1$ into sets $X_0, X_1 \in \mathcal{B}$ such that $\mu|_{X_0} \perp \nu|_{X_0}$ and $\mu|_{X_1} \ll \nu|_{X_1}$.

HINT: First assume μ and ν are finite and do a $\frac{1}{2}$ μ-measure exhaustion of ν-null sets to get X_0.

(b) Deduce that there is a partition $X = X_0 \sqcup X_1$ into sets $X_0, X_1 \in \mathcal{B}$ such that $\mu|_{X_0} \perp \nu|_{X_0}$ and $\mu|_{X_1} \sim \nu|_{X_1}$. Show that this partition is unique up to $(\mu + \nu)$-null sets, i.e. if $X = \tilde{X}_0 \sqcup \tilde{X}_1$ is another such partition, then $X_i \triangle \tilde{X}_i$ is $(\mu + \nu)$-null for each $i = 0, 1$.

(c) Suppose that $\mu \ll \nu$ and prove that the Radon–Nikodym derivative $\frac{d\mu}{d\nu}$ is unique up to null sets, i.e. if f, g are $\mathcal{B}$-measurable non-negative functions such that $g \, d\nu = \mu = f \, d\nu$, then $f = g$ a.e.

Definition. Let $f : X \to Y$, where X is a topological space and (Y, d) is a metric space. Define the functions $\mathrm{osc}_f : X \to [0, \infty]$ by

$$\mathrm{osc}_f(x) := \inf\{\mathrm{diam}_d(f(U)) : x \in U \subseteq X \text{ open}\},$$

where $f(U) \subseteq Y$ is the f-image of the set U and $\operatorname{diam}_d(Y') := \sup\{d(y_0, y_1) : y_0, y_1 \in Y'\}$ for each $Y' \subseteq Y$. Note that the set $C_f := \left\{x \in X : \operatorname{osc}_f(x) = 0\right\}$ is precisely the set of points at which f is continuous, so we call C_f the **set of continuity points** of f.

4. [*Optional, but read it*] Let $f : X \to Y$, where X is a topological space and (Y, d) is a metric space.

   (a) Prove that the set $\left\{x \in X : \operatorname{osc}_f(x) < \alpha\right\}$ is open for each $\alpha \in [0, \infty]$.

   (b) Deduce that $\operatorname{osc}_f : X \to [0, \infty]$ is a Borel function and C_f is G_δ (even if f is far from being Borel).

   (c) Conclude that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at rationals but discontinuous at irrationals.

   (d) Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at irrationals but discontinuous at rationals.

5. **Riemann integration.** Let λ be the Lebesgue measure on $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a bounded function, $a < b \in \mathbb{R}$. For a finite partition $\mathcal{P}$ of $[a, b]$ into intervals, let $\|\mathcal{P}\|$ denote its **mesh**, i.e. maximum length of an interval in $\mathcal{P}$. Let $\underline{f}_{\mathcal{P}} := \sum_{I \in \mathcal{P}} a_I \mathbb{1}_I$ and $\overline{f}_{\mathcal{P}} := \sum_{I \in \mathcal{P}} A_I \mathbb{1}_I$, where $a_I := \inf_{x \in I} f(x)$ and $A_I := \sup_{x \in I} f(x)$. Fix a sequence $(\mathcal{P}_n)$ of finite partitions of $[a, b]$ into intervals such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$, and $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$.

   (a) Prove that the sequences $(\underline{f}_{\mathcal{P}_n})$ and $(\overline{f}_{\mathcal{P}_n})$ are monotone, hence the limits $\underline{f} := \lim_n \underline{f}_{\mathcal{P}_n}$ and $\overline{f} := \lim_n \overline{f}_{\mathcal{P}_n}$ exist and are Borel functions such that $\underline{f} \leqslant f \leqslant \overline{f}$.

   (b) Recall the definition of a Riemann integrable function, and prove that f is Reimann integrable if and only if $\int \underline{f}\, d\lambda = \int \overline{f}\, d\lambda$ if and only if $\underline{f} = \overline{f}$ a.e.

   Hint: For the first equivalence, note that $\int \underline{f}\, d\lambda$ and $\int \overline{f}\, d\lambda$ are exactly the limits of the lower and upper sums of the partition $\mathcal{P}_n$.

   (c) Deduce that if f is Riemann integrable then it is Lebesgue measurable and its Riemann integral $\int_a^b f(t) dt$ is equal to its Lebesgue integral $\int_{[a,b]} f\, d\lambda$.

   (d) Also prove that f is Riemann integrable if and only if it is continuous at a.e. point in $[a, b]$, i.e. the set C_f of continuity points of f is conull in $[a, b]$.

   Hint: This question is partially answered in Folland's Theorem 2.28 on page 57, and I don't mind if you read its proof.

6. Let μ be a Borel measure on $\mathbb{R}$ that is finite on bounded intervals. Let $f_\mu : \mathbb{R} \to \mathbb{R}$ be any function such that $\mu((a, b]) = f_\mu(b) - f_\mu(a)$; for example, $f_\mu(x) := \mu((0, x])$ for $x \geqslant 0$, and $f_\mu(x) := -\mu((x, 0])$ for $x < 0$. Suppose that f_μ is differentiable and f'_μ is continuous, and prove that $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = f'_\mu$.

7. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\mathcal{C} \subseteq \mathcal{B}$ be a sub-σ-algebra witnessing the σ-finiteness of μ, i.e. $X = \bigcup_{n \in \mathbb{N}} C_n$ where each $C_n \in \mathcal{C}$ and $\mu(C_n) < \infty$. Thus, the restriction $\nu := \mu|_{\mathcal{C}}$ is σ-finite. [1]

(a) Prove that for each μ-measurable (i.e. $\mathcal{B}$+null) $f \in L^1(\mu)$, there is a ν-measurable (i.e. $\mathcal{C}$+null) $f_{\mathcal{C}} \in L^1(\mu)$ such that $\int_C f \, d\mu = \int_C f_{\mathcal{C}} \, d\mu$ for each $C \in \mathcal{C}$. This function $f_{\mathcal{C}}$ is unique up to a μ-null set (prove this as well) and it is called the **conditional expectation** of f with respect to the sub-σ-algebra $\mathcal{C}$.

Hint: First suppose that $f \geqslant 0$, and consider the measure $\nu_f := \mu_f|_{\mathcal{C}}$ on $\mathcal{C}$, where $\mu_f(B) := \int_B f \, d\mu$. Observe that $\nu_f \ll \nu$ so $\frac{d\nu_f}{d\nu}$ exists.

Caution: For a ν-measurable function $g : X \to \mathbb{R}$, the integrals $\int g \, d\nu$ and $\int g \, d\mu$ have different definitions (one uses ν-measurable simple functions, the other one μ-measurable). However, thanks to the monotone convergence theorem, these integrals are equal. You have to prove this since it is used in the solution.[2]

(b) To get a handle on conditional expectation, let $\mathcal{C}$ be the σ-algebra generated by a countable partition $\mathcal{P} \subseteq \mathcal{B}$ of X and compute $f_{\mathcal{C}}$ explicitly in terms of f and $\mathcal{P}$.

Hint: In this case, $f_{\mathcal{C}}$ is a countable linear combination of indicator functions.

8. Consider the space $\mathbb{R}^d$ with Lebesgue measure λ and let $r > 0$. Let A_r be the averaging operator on L^1 defined by $A_r f(x) := \frac{\int_{B_r(x)} f \, d\lambda}{\lambda(B_r(x))}$, where $B_r(x)$ is the (open) ball of radius r centered at x in the d_∞ metric.

(a) Prove the local-global bridge lemma: $\int f \, d\lambda = \int A_r f \, d\lambda$ for all $f \in L^1$. In particular, A_r is an L^1**-contraction**, i.e. $\|A_r f\|_1 \leqslant \|f\|_1$ for all $f \in L^1$, and hence $A_r : L^1 \to L^1$.

(b) Prove that for each $f \in L^1$, the function $(r, x) \mapsto A_r f(x)$ is continuous as a function $(0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, i.e. it is jointly continuous in (r, x).

[1] As pointed out by some of you (thanks!), this assumption is necessary: the sub-σ-algebra of null and conull sets of an infinite σ-finite measure would be a counter-example.

[2] Thanks to Owen Rodgers for asking about this.